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Conductivity of a Hot Electron Gas

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The dissipative part of the high-frequency wave-number-dependent conductivity tensor of a hot, dilute electron gas near equilibrium is calculated exactly to terms proportional to the plasma parameter k_D^3/n (the inverse of the number of particles in a Debye sphere), in the limit of classical statistics. The calculation includes high-frequency collective dynamic screening effects consistently to this order. These collective effects have an important effect on the frequency dependence of the conductivity for frequencies greater than about twice the electron-plasma frequency.

I. INTRODUCTION

THIS paper is concerned with the calculation of the dissipative part of the high-frequency wave-number-dependent collisional conductivity of a hot, dilute electron gas (in a uniform positive background) near equilibrium.

The problem has attracted considerable theoretical interest. Several calculations have been presented[1–5] of the collisional damping rate of plasma waves, which is proportional to the dissipative part of the conductivity near the plasma frequency. However, no two of them agree because they make different approximations in treating the collective behavior of the electron gas. The present calculation includes the collective effects consistently so that in the limit of high temperatures ($kT \gg$rydberg) the result is exact to first order in the weak coupling parameter k_D^3/n.

Our calculation shows that the detailed frequency dependence of the conductivity is effected by high-frequency collective effects. At twice the plasma frequency there is very slight inflection in the conductivity curve which can be interpreted as resulting from the excitation of two plasmons. Because there is no actual resonance near $\omega \simeq 2\omega_p$, it is shown from this calculation that frequency shifts in incoherent scattering of radiation from a uniform plasma at the second harmonic of the plasma frequency will not be observable.

The electron-electron collisional conductivity which is proportional to k^2 for small wave numbers k is generally not of much practical interest for real plasmas since the contribution from electron-ion collisions is independent of k in this limit and also has a k^2 term of comparable magnitude to the electron-electron contribution. However, the frequency dependence of the electron-electron contribution is different than the ion-electron contribution in the frequency region near $2\omega_p$.

The high-frequency collective effect arises from the perturbation of the screening electrons by the high-frequency incident radiation. The same type of collective effect was first pointed out by Dawson and Oberman[6] in the calculation of the electron-ion collisional conductivity where it produced a weak resonance and inflection at $\omega = \omega_p$. This collective effect is not found in the low-frequency kinetic equations of the Balescu type.[7,8]

The method to be used is formally similar to that of a previous calculation[9] of the electron-ion collisional conductivity. The various possible processes are represented by Feynman diagrams. In Ref. 9 we presented a convenient set of rules for translating the diagrams into integrals. For convenience these rules are listed in the Appendix in a form suitable for this problem. As we have tried to emphasize before, this method has the advantage of relating directly to quantities of physical interest and deals with the microscopic processes in a very intuitive and explicit way.

The calculation in Sec. III includes exactly the electron-electron collisional effects in a classical plasma[10] to lowest order in the plasma parameter $\lambda = k_D^3/n$ (k_D the Debye wave number, $k_D^2 = 4\pi e^2 n/kT$, n the number density). It is assumed that the temperature is high enough so that collisions can be treated in Born approximation [i.e., $e^2/\hbar(kT/m)^{1/2} \ll 1$] which introduces a

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[9] D. F. Du Bois, V. Gilinsky, and M. G. Kivelson, Phys. Rev. **129**, 2376 (1963).
[10] The term "classical" means here that the gas obeys Boltzmann statistics [i.e., $\hbar^3/(mkT)^{3/2} \ll n^{-1}$] and that quantum-mechanical interference effects can be neglected [i.e., the thermal deBroglie wavelength is much less than the Debye length, $\hbar(4\pi e^2 n/m)^{1/2} \ll kT$].

natural short-range cutoff at the thermal deBroglie wavelength. At lower temperatures the usual classical approximation will be made of using the distance of closest approach e^2/kT as the short range cutoff. We will consider incident only frequencies much higher than the collision frequencies (of order $\lambda\omega_p$) and wave numbers small compared to the Debye wave number. An important feature of the calculation is the demonstration of the necessity of including the perturbation of the screening electrons to obtain a consistent result at high frequencies which has the correct proportionality to k^2 (for small k) which is demanded by conservation of total current.

In Sec. IV we discuss the frequency dependence of our results. At $\omega=\omega_p$ we compare our results with the recent work of Wu and Klevans.[3]

We find weak inflection in the conductivity curves at $2\omega_p$ due to double plasmon production. There is no sharp resonance at this frequency due to the dispersion and damping of the plasmons. We conclude this section by commenting on the application of these results to the line shape of incoherently scattered radiation from a plasma[11] near frequencies displaced by $\pm 2\omega_p$ from the incident frequencies.

II. GENERAL BACKGROUND

The starting point for our computation of the local conductivity[12] is the general expression, obtained from Eqs. (3.5) and (4.12) of Ref. 9 and Appendix A of Ref. 4, which relates the conductivity of matrix elements of the Heisenberg current operator $J_i(t)$.

$$4\pi\, \mathrm{Im}\sigma_{ij}(k,\omega)=(1/2\omega)\sum_{nm}{}' \rho_n\langle n|J_i(0)|m\rangle \times\langle m|J_j(0)|n\rangle(1-e^{-\beta\hbar\omega})(2\pi)^3 \times\delta^3(\hbar\mathbf{k}-\mathbf{P}_m+\mathbf{P}_n)2\pi\delta(\hbar\omega-E_m+E_n). \quad (2.1)$$

The dentisy matrix is

$$\rho_n=e^{\beta\Omega}e^{-\beta(E_n-\mu N_n)},\quad e^{-\beta\Omega}=\sum_n e^{-\beta(E_n-\mu N_n)}, \quad (2.2)$$

where μ is the chemical potential and, of course, $\mathbf{P}_n$, E_n, and N_n are the momentum, energy, and number of particles in the state n.

Equation (1) is similar to the golden rule of time-dependent perturbation theory, and it is shown in Ref. 9 how the result can be calculated in terms of a diagrammatic expansion in a coupling parameter. The prime on the summation indicates that in perturbation theory only *proper* diagrams are to be considered in calculating the *local* conductivity. (This is the point discussed in Appendix A of Ref. 4.) We can then give a simple prescription for directly calculating $\mathrm{Im}\sigma_{ij}(k,\omega)$.

Units

We shall use the same units as in Ref. 9, $\beta^{-1}=kT$ is the natural unit of energy, $\omega_p=(4\pi e^2n/m)^{1/2}$ is the natural unit of frequency [and so $\hbar$ will be measured in units of $(\beta\omega_p)^{-1}$], $p_T=(m/\beta)^{1/2}$ is the natural unit of momentum, and $k_D=(4\pi e^2n\beta)^{1/2}$ is the natural unit of wave number, and the coupling constant which arises naturally is $\lambda=k_D{}^3/n$.

These units are especially convenient for doing classical problems since it is easy to see the order of diagrams, both in λ and $\hbar$. In the limit of very high temperatures we are still left with a term proportional to $\ln(\hbar)$.

It follows from the considerations in Refs. 4 and 9 and Langer's proofs[13] on the analytic properties of expressions of the form of Eq. (2.1) in quantum statistical mechanics that $4\pi\,\mathrm{Im}\sigma_{ij}$ can be computed in perturbation theory in terms of unperturbed quantum statistical states α and β.

These states are understood relative to the state of complete thermal equilibrium which is to be considered as sort of a vacuum state. They are described in terms of the number of *particles* of given momenta (upward lines) representing an excess of one particle in each momentum state relative to the equilibrium population and the number of *holes* (downward lines) representing a depletion of one particle in the given state relative to equilibrium. The formula for $\mathrm{Im}\sigma_{ij}$ is similar in form to Eq. (2.1):

$$4\pi\,\mathrm{Im}\sigma_{ij}(k,\omega)=(1/2\omega)\sum_{\alpha\beta}{}' \omega_\alpha J_i(\alpha,\beta;k,\omega)J_i^*(\alpha,\beta;k,\omega) \times(1-e^{-\beta\hbar\omega})(2\pi\hbar)^3\delta^3(\hbar\mathbf{k}-\mathbf{P}_\beta+\mathbf{P}_\alpha) \times 2\pi\delta(\hbar\omega-E_\beta+E_\alpha), \quad (2.3)$$

where $\mathbf{P}_\alpha(E_\alpha)$ is the sum of particle momenta (energies) minus hole momenta (energies) in the unperturbed state α. ω_α is the statistical weighting factor for the initial state α described below in rule 7. The current amplitudes J_i are computed in perturbation theory according to the set of rules given in the Appendix.

The final state can consist of a single pair which corresponds to collisionless Landau damping and its virtual corrections.[9] A final state of two pairs can be obtained if collisions between particles are taken into account. The basic collision process taking into account screening in the random-phase approximation is shown in Fig. 1(a). The initial photon must be connected to this final state in all equivalent ways. We must attach the photon in turn to *each* particle line *including* the lines in the polarization loops. This is shown for a prototype diagram in Fig. 1(b) where the ×'s represent the possible places for attaching a photon line. For each particular photon insertion, additional polarization parts can be inserted to fully screen the Coulomb lines. The

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[12] Our definition of the conductivity is such that $j_i(k,\omega)=-i\sigma_{ij}\times(k,\omega)E_j(k,\omega)$ so real and imaginary parts are interchanged from the usual definition. We shall compute only the imaginary part of the conductivity. The real part corresponds to the polarizability and is found from the imaginary part via a familiar dispersion relation.

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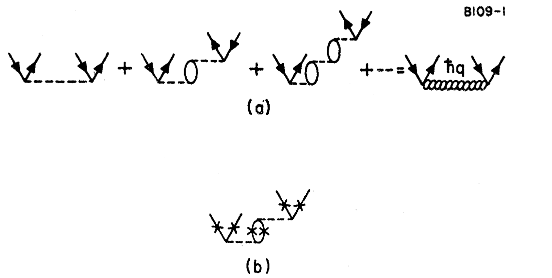

FIG. 1. (a) Basic diagrams for the dynamically screened collisions. (b) Typical diagram showing possible locations, indicated by ×'s for attachment of an external photon line.

result is the complete set of diagrams in Fig. 2. Note again that in any diagram a closed loop can be inserted along with an additional interaction line without changing the order in λ. This is because each closed loop will have at least one factor of $1/\lambda$ arising from the e^{μ} of Eqs. (A2) and (A10) which cancels the additional factor of λ from the extra interaction line. This principle operates in forming the screened interaction propagator and in the diagrams with the internal loops of Fig. 2.

It is easy to see that all other diagrams leading to the same final state must be higher powers in λ since one must add more interaction lines than closed loops. We can neglect such contributions to obtain the leading term in λ.

Before turning to the calculation let us make a few comments concerning these diagrams. The resonance at $\omega=2\omega_p$ comes, of course, from the plasmon resonances in the two screened interaction propagators in Figs. 2(e) and 2(f). Since this resonant effect is of greatest interest, it would be tempting to consider these two diagrams as dominant and neglect the other four. However, we shall see that even near $\omega=2\omega_p$ there is an important cancellation between diagrams (e), (f) and diagrams (a), (b), (c), (d) which must be taken into account to get the proper k^2 proportionality of the conductivity for small k.

It follows from the symmetries of the system that the tensor conductivity can be decomposed into longitudinal and transverse parts

$$\sigma_{ij}(k,\omega)=(k_ik_j/k^2)\sigma_L(k,\omega)+[\delta_{ij}-(k_ik_j/k^2)]\sigma_T(k,\omega). \quad (2.4)$$

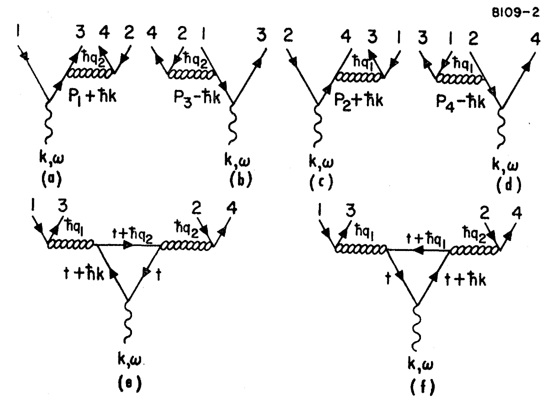

FIG. 2. Diagrams of order λ contributing to collisional conductivity.

The functions σ_L and σ_T can be projected out of σ_{ij} by using unit polarization vectors $\hat e^{(\alpha)}$ where $\hat e^{(0)}=\hat k$ for the longitudinal case and $\hat e^{(1,2)}$ are perpendicular to $\hat k$ and to each other in the transverse case. The well-known sum over transverse polarizations takes the form

$$\sum_{\alpha=1,2}\hat e_i{}^{\alpha}\hat e_j{}^{\alpha}=\delta_{ij}-(k_ik_j/k^2). \quad (2.5)$$

Using these definitions we can write

$$\sigma_L(k,\omega)=\hat e_i{}^{(0)}\hat e_j{}^{(0)}\sigma_{ij}(k,\omega)=\hat k_i\hat k_j\sigma_{ij}(k,\omega), \quad (2.6)$$

$$\sigma_T(k,\omega)=\tfrac{1}{2}\sum_{\alpha=1,2}\hat e_i{}^{\alpha}\hat e_j{}^{\alpha}\sigma_{ij}(k,\omega). \quad (2.7)$$

In the following section we will apply these rules to the calculation of the collisional conductivity of electron gas from the diagrams of Fig. 2. In Sec. IV we will discuss these results and their application to a realistic two-component plasma.

III. CALCULATION

From the rules of the previous section and Fig. 2, the following expression for the dissipative part of the conductivity due to electron-electron collisions ($m=1$) can immediately be set down:

$$4\pi\hat e_i\hat e_j\,\mathrm{Im}\sigma_{ij}(k,\omega)=\frac{1}{4\omega}\int\frac{d^3p_1}{(2\pi\hbar)^3}\int\frac{d^3p_2}{(2\pi\hbar)^3}\int\frac{d^3p_3}{(2\pi\hbar)^3}\int\frac{d^3p_4}{(2\pi\hbar)^3}\cdot\hbar^6((2\pi)^3/\lambda^2)e^{-\frac{1}{2}p_1{}^2}e^{-\frac{1}{2}p_2{}^2}(1-e^{-\hbar\omega})|\mathbf{J}\cdot\hat e|^2$$
$$\times(2\pi\hbar)^3\delta^3(\hbar\mathbf{k}+\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)2\pi\delta(\hbar\omega+\xi_1+\xi_2-\xi_3-\xi_4), \quad (3.1)$$

where $\xi_1=\xi(p_1)$, etc. The analytically continued (see rule 8) current amplitudes are given by

$$\mathbf{J}\cdot\hat e=i\lambda V_s{}^+(\mathbf{p}_4-\mathbf{p}_2,\xi_4-\xi_2)\left[\frac{\lambda^{1/2}(\mathbf{p}_1+\frac{1}{2}\hbar\mathbf{k})\cdot\hat e}{\hbar\omega-\xi(\mathbf{p}_1+\hbar\mathbf{k})+\xi_1}-\frac{\lambda^{1/2}(\mathbf{p}_3-\frac{1}{2}\hbar\mathbf{k})\cdot\hat e}{\hbar\omega+\xi(\mathbf{p}_3-\hbar\mathbf{k})-\xi_3}\right]$$
$$+i\lambda V_s{}^+(\mathbf{p}_3-\mathbf{p}_1,\xi_3-\xi_1)\left[\frac{\lambda^{1/2}(\mathbf{p}_2+\frac{1}{2}\hbar\mathbf{k})\cdot\hat e}{\hbar\omega-\xi(\mathbf{p}_2+\hbar\mathbf{k})+\xi_2}-\frac{\lambda^{1/2}(\mathbf{p}_4-\frac{1}{2}\hbar\mathbf{k})\cdot\hat e}{\hbar\omega+\xi(\mathbf{p}_4-\hbar\mathbf{k})-\xi_4}\right]$$
$$-\lambda^2V_s{}^+(\mathbf{p}_4-\mathbf{p}_2,\xi_4-\xi_2)V_s(\mathbf{p}_3-\mathbf{p}_1,\xi_3-\xi_1)\hat e\cdot\mathbf{T}(\mathbf{p}_3-\mathbf{p}_1,\xi_3-\xi_1;\mathbf{p}_4-\mathbf{p}_2,\xi_4-\xi_2;k,\omega), \quad (3.2)$$

where $\mathbf{T}$ is the amplitude arising from the triangular closed loops in Fig. 2. Defining the momentum and energy-transfer variables

$$\hbar\mathbf{q}_1=\mathbf{p}_3-\mathbf{p}_1,\quad \hbar u_1=\xi_3-\xi_1=\hbar\mathbf{q}_1\cdot\mathbf{p}_1+(\hbar^2/2)q_1^2,$$
$$\hbar\mathbf{q}_2=\mathbf{p}_4-\mathbf{p}_2,\quad \hbar u_2=\xi_4-\xi_2=\hbar\mathbf{q}_2\cdot\mathbf{p}_2+(\hbar^2/2)q_2^2, \tag{3.3}$$

we have

$$\hat{e}\cdot\mathbf{T}^+(q_1,u_1;q_2,u_2;k,\omega)=i\lambda^{1/2}\hat{e}\cdot\boldsymbol{\tau}^+(q_1,u_1;k,\omega)+i\lambda^{1/2}\hat{e}\cdot\boldsymbol{\tau}^+(q_2,u_2;k,\omega), \tag{3.4}$$

where the two functions $\boldsymbol{\tau}$ on the right-hand side arise from the two directions of the closed loop in Figs. 2 (e) and (f), respectively. These functions are the analytic continuations of

$$\hat{e}\cdot\boldsymbol{\tau}'(q_1,q_{10};k,k_0)=\int\frac{d^3t}{(2\pi\hbar)^3}\hat{e}\cdot(\mathbf{t}+\tfrac{1}{2}\hbar\mathbf{k})\sum_{t_0}\frac{1}{(t_0-\xi(\mathbf{t}))(t_0+k_0-\xi(\mathbf{t}+\hbar\mathbf{k}))(t_0+q_{10}-\xi(\mathbf{t}+\hbar\mathbf{q}_1))} \tag{3.5}$$

from the discrete values $k_0=i\pi n/\beta$, $q_{10}=i\pi n_1/\beta$, where n and n_1 are even integers to continuous values $\hbar(\omega+i\epsilon)$ and $\hbar(u_1+i\epsilon)$ in the upper complex half-planes. The sum is over $t_0=i\pi\nu/\beta$, where ν is an even integer for bosons and odd for fermions. The sum is easily carried out by converting to a contour integral with the result

$$\hat{e}\cdot\boldsymbol{\tau}'(q_1,q_{10};k,k_0)=-\int\frac{d^3t}{(2\pi\hbar)^3}\frac{(\mathbf{t}+\frac{1}{2}\hbar\mathbf{k})\cdot\hat{e}}{k_0-\xi(\mathbf{t}+\hbar\mathbf{k})+\xi(\mathbf{t})}\left[\frac{f(\mathbf{t})-f(\mathbf{t}+\hbar\mathbf{q}_1)}{q_{10}+\xi(\mathbf{t})-\xi(\mathbf{t}+\hbar\mathbf{q}_1)}-\frac{f(\mathbf{t}+\hbar\mathbf{k})-f(\mathbf{t}+\hbar\mathbf{q}_1)}{q_{10}-k_0+\xi(\mathbf{t}+\hbar\mathbf{k})-\xi(\mathbf{t}+\hbar\mathbf{q}_1)}\right]. \tag{3.6}$$

Now going back to Eq. (3.1) the momentum conserving δ function can be trivially integrated out by making the substitutions

$$\mathbf{q}_1=\mathbf{q}+\tfrac{1}{2}\mathbf{k},\quad \mathbf{q}_2=-\mathbf{q}+\tfrac{1}{2}\mathbf{k}. \tag{3.7}$$

The definitions of Eq. (3.3) for u_1 and u_2 can be introduced by inserting delta functions and Eq. (3.1) becomes

$$4\pi\hat{e}_i\hat{e}_j\,\mathrm{Im}\sigma_{ij}(k,\omega)=\frac{1}{4}\left(\frac{1-e^{-\hbar\omega}}{\hbar\omega}\right)\frac{(2\pi)^4}{\lambda^2}\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3p_1}{(2\pi)^3}\int\frac{d^3p_2}{(2\pi)^3}e^{-\frac{1}{2}(p_1^2+p_2^2)}\int_{-\infty}^{\infty}du_1\int_{-\infty}^{\infty}du_2\delta(\omega-u_1-u_2)$$
$$\times\delta[u_1-\mathbf{p}_1\cdot(\mathbf{q}+\tfrac{1}{2}\mathbf{k})-\tfrac{1}{2}\hbar(\mathbf{q}+\tfrac{1}{2}\mathbf{k})^2]\delta[u_2+\mathbf{p}_2\cdot(\mathbf{q}-\tfrac{1}{2}\mathbf{k})-\tfrac{1}{2}\hbar(\mathbf{q}-\tfrac{1}{2}\mathbf{k})^2]|\mathbf{J}\cdot\hat{e}|^2, \tag{3.8}$$

with

$$-i(\mathbf{J}\cdot\hat{e})=-\lambda^{3/2}V_s^+(\hbar\mathbf{q}+\tfrac{1}{2}\hbar\mathbf{k},\hbar u_1)\frac{1}{\hbar}\left[\frac{(\mathbf{p}_2+\frac{1}{2}\hbar\mathbf{k})\cdot\hat{e}}{\omega-\mathbf{k}\cdot\mathbf{p}_2-\frac{1}{2}\hbar k^2}-\frac{(\mathbf{p}_2+\hbar\mathbf{q})\cdot\hat{e}}{\omega-\mathbf{k}\cdot\mathbf{p}_2-\hbar\mathbf{k}\cdot\mathbf{q}}\right]+\lambda^{3/2}V_s^+(-\hbar\mathbf{q}+\tfrac{1}{2}\hbar\mathbf{k},\hbar u_2)(1/\hbar)$$
$$\times\left[\begin{array}{c}\text{same term with}\\ \mathbf{q}\to-\mathbf{q},\ \mathbf{p}_2\to\mathbf{p}_1\end{array}\right]-\lambda^{5/2}V_s^+(\hbar\mathbf{q}+\tfrac{1}{2}\hbar\mathbf{k},\hbar u_1)V_s^+(-\hbar\mathbf{q}+\tfrac{1}{2}\hbar\mathbf{k},\hbar u_2)$$
$$\times[\hat{e}\cdot\boldsymbol{\tau}(\mathbf{q}+\tfrac{1}{2}\mathbf{k},u_1;k,\omega)+\hat{e}\cdot\boldsymbol{\tau}(-\mathbf{q}+\tfrac{1}{2}\mathbf{k},u_2;k,\omega)]. \tag{3.9}$$

It is convenient next to take the limit $\hbar\to 0$; Rembering that in this limit $V_s^+(\hbar q,\hbar u)=V_s^+(q,u)$

$$-i\mathbf{J}^c\cdot\hat{e}=i\lim_{\hbar\to 0}\mathbf{J}\cdot\hat{e}=\lambda^{3/2}V_s^+(\mathbf{q}+\tfrac{1}{2}\mathbf{k},u_1)V_s^+(\mathbf{q}-\tfrac{1}{2}\mathbf{k},u_2)\left\{V_s^{-1}\left(\mathbf{q}-\frac{\mathbf{k}}{2},u_2\right)\left[\frac{\frac{1}{2}(\mathbf{k}\cdot\hat{e})-(\hat{e}\cdot\mathbf{q})}{\omega-\mathbf{k}\cdot\mathbf{p}_2}-\frac{(\mathbf{k}\cdot\mathbf{q}-k^2)(\mathbf{p}_2\cdot\hat{e})}{(\omega-\mathbf{k}\cdot\mathbf{p}_2)^2}\right]\right.$$
$$+V_s^{-1}\left(\mathbf{q}+\frac{\mathbf{k}}{2},u_1\right)\left[\frac{\frac{1}{2}(\mathbf{k}\cdot\hat{e})+(\hat{e}\cdot\mathbf{q})}{\omega-\mathbf{k}\cdot\mathbf{p}_1}+\frac{(\mathbf{k}\cdot\mathbf{q}+k^2)}{(\omega-\mathbf{k}\cdot\mathbf{p}_1)^2}(\mathbf{p}_1\cdot\hat{e})\right]$$
$$\left.-\lambda^{1/2}\lim_{\hbar\to 0}\left[\hat{e}\cdot\boldsymbol{\tau}^+\left(\mathbf{q}+\frac{\mathbf{k}}{2},u_1;k,\omega\right)+\hat{e}\cdot\boldsymbol{\tau}^+\left(-\mathbf{q}+\frac{\mathbf{k}}{2},u_2;k,\omega\right)\right]\right\}. \tag{3.10}$$

From Eq. (3.6) after analytic continuation

$$\hat{e}\cdot\boldsymbol{\tau}^+\left(\mathbf{q}+\frac{\mathbf{k}}{2},u_1;k,\omega\right)=-\frac{1}{\hbar^2}\int\frac{d^3t}{(2\pi\hbar)^3}\frac{(\mathbf{t}+\frac{1}{2}\hbar\mathbf{k})\cdot\hat{e}}{\omega-\mathbf{k}\cdot(\mathbf{t}+\frac{1}{2}\hbar\mathbf{k})}$$
$$\times\left[\frac{f(\mathbf{t})-f(\mathbf{t}+\hbar\mathbf{q}+\frac{1}{2}\hbar\mathbf{k})}{u_1-(\mathbf{q}+\frac{1}{2}\mathbf{k})\cdot(\mathbf{t}+\frac{1}{2}\hbar\mathbf{q}+\frac{1}{4}\hbar\mathbf{k})}+\frac{f(\mathbf{t}+\hbar\mathbf{k})-f(\mathbf{t}+\hbar\mathbf{q}+\frac{1}{2}\hbar\mathbf{k})}{u_2-\mathbf{k}\cdot(\mathbf{t}+\frac{1}{2}\hbar\mathbf{k})+(\mathbf{q}+\frac{1}{2}\mathbf{k})\cdot(\mathbf{t}+\frac{1}{2}\hbar\mathbf{q}+\frac{1}{4}\hbar\mathbf{k})}\right], \tag{3.11}$$

where the energy conservation condition $\omega = u_1 + u_2$ has been used, and where ω, u_1, u_2 are understood to have small positive imaginary parts.

This equation can be made more symmetrical looking by making the substitutions $\mathbf{t} \to \mathbf{t} - \frac{1}{2}\hbar(\mathbf{q} + \frac{1}{2}\mathbf{k})$ in the first term in square brackets and $\mathbf{t} + \hbar\mathbf{k} \to \mathbf{t} - \frac{1}{2}\hbar(\mathbf{q} - \frac{1}{2}\mathbf{k})$ in the second term. The result can be written in the form

$$\hat{e}\cdot\boldsymbol{\tau}(\mathbf{q}+\tfrac{1}{2}\mathbf{k}, u_1; k,\omega) = \hat{e}\cdot\boldsymbol{\theta}(\mathbf{q},u_1; k,\omega; \hbar) + \hat{e}\cdot\boldsymbol{\theta}(-\mathbf{q},u_2; k,\omega; -\hbar), \tag{3.12}$$

where

$$\hat{e}\cdot\boldsymbol{\theta}(\mathbf{q},u_1; k,\omega; \hbar) = -\frac{1}{\hbar^2}\int \frac{d^3t}{(2\pi\hbar)^3} \frac{\hat{e}\cdot[\mathbf{t}-\frac{1}{2}\hbar(\mathbf{q}-\frac{1}{2}\mathbf{k})]}{\omega - \mathbf{k}\cdot(\mathbf{t}-\frac{1}{2}\hbar(\mathbf{q}-\frac{1}{2}\mathbf{k}))} \cdot \left[\frac{f(\mathbf{t}-\frac{1}{2}\hbar(\mathbf{q}+\frac{1}{2}\mathbf{k})) - f(\mathbf{t}+\frac{1}{2}\hbar(\mathbf{q}+\frac{1}{2}\mathbf{k}))}{u_1 - (\mathbf{q}+\frac{1}{2}\mathbf{k})\cdot\mathbf{t}}\right]. \tag{3.13}$$

From Eq. (3.12) it follows that

$$\lim_{\hbar\to 0}\hat{e}\cdot[\boldsymbol{\tau}(\mathbf{q}+\tfrac{1}{2}\mathbf{k}, u_1; k,\omega) + \boldsymbol{\tau}(-\mathbf{q}+\tfrac{1}{2}\mathbf{k}; u_2; k,\omega)] \tag{3.14}$$

$$= \lim_{\hbar\to 0}\hat{e}\cdot[\boldsymbol{\theta}(\mathbf{q},u_1; k,\omega; \hbar) + \boldsymbol{\theta}(-\mathbf{q},u_2; k,\omega; -\hbar) + \boldsymbol{\theta}(-\mathbf{q},u_2; k,\omega; \hbar) + \boldsymbol{\theta}(\mathbf{q},u_1; k,\omega; -\hbar)]. \tag{3.15}$$

Since the leading term in $\hat{e}\cdot\boldsymbol{\theta}$, as $\hbar \to 0$, goes as $\hbar^{-1}$, this term cancels out in the limit and the remaining term is independent of $\hbar$

$$\lim_{\hbar\to 0}\hat{e}\cdot[\boldsymbol{\tau}(\mathbf{q}+\tfrac{1}{2}\mathbf{k}, u_1; k,\omega) + \boldsymbol{\tau}(-\mathbf{q}+\tfrac{1}{2}\mathbf{k}, u_2; k,\omega)]$$

$$= \int \frac{d^3t}{(2\pi\hbar)^3}\left[\frac{\hat{e}\cdot(\mathbf{q}-\frac{1}{2}\mathbf{k})}{\omega - \mathbf{k}\cdot\mathbf{t}} + \frac{(\hat{e}\cdot\mathbf{t})\mathbf{k}\cdot(\mathbf{q}-\frac{1}{2}\mathbf{k})}{(\omega-\mathbf{k}\cdot\mathbf{t})^2}\right]\frac{\mathbf{t}\cdot(\mathbf{q}+\frac{1}{2}\mathbf{k})}{u_1 - \mathbf{t}\cdot(\mathbf{q}+\frac{1}{2}\mathbf{k})} f(\mathbf{t}) + (\mathbf{q}\to -\mathbf{q},\ u_1 \to u_2). \tag{3.16}$$

The case $k \ll 1$ (i.e., $k \ll k_D$) is of greatest interest so the next step is to expand in powers of k. The right-hand side of Eq. (3.16) becomes

$$\int \frac{d^3t}{(2\pi\hbar)^3}\left[\frac{(\hat{e}\cdot\mathbf{q})}{\omega} - \frac{(\hat{e}\cdot\mathbf{k})}{2\omega} + \frac{(\hat{e}\cdot\mathbf{q})(\mathbf{k}\cdot\mathbf{t})}{\omega^2} + \frac{(\hat{e}\cdot\mathbf{t}(\mathbf{k}\cdot\mathbf{q})}{\omega^2}\right]\frac{\mathbf{t}\cdot\mathbf{q} f(\mathbf{t})}{u_1 - \mathbf{t}\cdot\mathbf{q}}$$

$$+ \int \frac{d^3t}{(2\pi\hbar)^3}\frac{(\hat{e}\cdot\mathbf{q})}{\omega}\left[\frac{1}{2}\frac{\mathbf{t}\cdot\mathbf{k}}{u_1 - \mathbf{t}\cdot\mathbf{q}} f(\mathbf{t}) + \frac{1}{2}\frac{\mathbf{t}\cdot\mathbf{q}\mathbf{k}\cdot\mathbf{t}f(\mathbf{t})}{(u_1 - \mathbf{t}\cdot\mathbf{q})^2}\right] + \begin{pmatrix}\mathbf{q}\to -\mathbf{q}\\ u_1 \to u_2\end{pmatrix}. \tag{3.17}$$

Using the Eq. (2.7) we see that

$$\lim_{\hbar\to 0} Q_0^+(\hbar q, \hbar u) = Q_0^+\left(\frac{u}{q}\right) = \lambda\int \frac{d^3t}{(2\pi\hbar)^3}\frac{\mathbf{t}\cdot\mathbf{q}}{u - \mathbf{t}\cdot\mathbf{q}} f(\mathbf{t}) = \int \frac{d^3t}{(2\pi)^{3/2}}\frac{\mathbf{t}\cdot\mathbf{q}}{u - \mathbf{t}\cdot\mathbf{q}} e^{-\frac{1}{2}t^2} \tag{3.18}$$

and

$$\frac{\partial}{\partial q^2} Q_0^+(q,u) = \frac{\lambda}{2q^2}\int \frac{d^3t}{(2\pi\hbar)^3}\left[\frac{\mathbf{t}\cdot\mathbf{q}}{(u - \mathbf{t}\cdot\mathbf{q})} + \frac{(\mathbf{t}\cdot\mathbf{q})^2}{(u - \mathbf{t}\cdot\mathbf{q})^2}\right] f(\mathbf{t}). \tag{3.19}$$

We can write Eq. (3.17) in the form

$$\lambda \lim_{\hbar\to 0}[\boldsymbol{\tau}(\mathbf{q}+\tfrac{1}{2}\mathbf{k}, u_1; k,\omega) + \boldsymbol{\tau}(-\mathbf{q}+\tfrac{1}{2}\mathbf{k}, u_2; k,\omega)] = (\hat{e}\cdot\mathbf{q})\frac{1}{\omega}\left[Q_0^+\left(\frac{u_1}{q}\right) - Q_0^+\left(\frac{u_2}{q}\right)\right] - (\hat{e}\cdot\mathbf{k})\frac{1}{2\omega}\left[Q_0^+\left(\frac{u_1}{q}\right) + Q_0^+\left(\frac{u_2}{q}\right)\right]$$

$$+ \frac{(\mathbf{k}\cdot\mathbf{q})(\hat{e}\cdot\mathbf{q})}{\omega}\left[\frac{\partial}{\partial q^2}Q_0^+\left(\frac{u_1}{q}\right) - \frac{\partial}{\partial q^2}Q_0^+\left(\frac{u_2}{q}\right)\right] + 2(\hat{e}\cdot\mathbf{q})\frac{\mathbf{k}\cdot\mathbf{q}}{\omega^2 q^2}\left[u_1 Q_0^+\left(\frac{u_1}{q}\right) + u_2 Q_0^+\left(\frac{u_2}{q}\right)\right] + O(k^2). \tag{3.20}$$

To complete the expansion of $(\mathbf{J}\cdot\hat{e})$ to linear terms in k, the first two terms in brackets in Eq. (3.10) must be expanded. Carrying this out and using Eq. (3.20) the complete result is

$$-i(\hat{e}\cdot\mathbf{J}^c) = -\lambda^{3/2} V_s^+(q,u_1) V_s^+(q,u_2)\left\{\left[Q_0^+\left(\frac{u_1}{q}\right) - Q_0^+\left(\frac{u_2}{q}\right)\right]\frac{(\hat{e}\cdot\mathbf{q})}{\omega} + \frac{2}{\omega}(\hat{e}\cdot\mathbf{q})(\mathbf{k}\cdot\mathbf{q}) + \frac{(\mathbf{k}\cdot\mathbf{q})}{\omega}(\hat{e}\cdot\mathbf{q})\right.$$

$$\times\left[\frac{\partial}{\partial q^2}Q_0^+\left(\frac{u_1}{q}\right) - \frac{\partial}{\partial q^2}Q_0^+\left(\frac{u_2}{q}\right)\right] + \left[q^2 + Q_0^+\left(\frac{u_1}{q}\right)\right]\left[\frac{1}{2\omega}(\hat{e}\cdot\mathbf{k}) + \frac{(\mathbf{k}\cdot\mathbf{q})(\hat{e}\cdot\mathbf{p}_1)}{\omega^2} + \frac{(\mathbf{k}\cdot\mathbf{p}_1)(\hat{e}\cdot\mathbf{q})}{\omega^2}\right]$$

$$\left. + \left[q^2 + Q_0^+\left(\frac{u_2}{q}\right)\right]\left[\frac{1}{2\omega}(\hat{e}\cdot\mathbf{k}) - \frac{(\mathbf{k}\cdot\mathbf{q})(\hat{e}\cdot\mathbf{p}_2)}{\omega^2} - \frac{(\mathbf{k}\cdot\mathbf{p}_2)(\hat{e}\cdot\mathbf{q})}{\omega^2}\right] + [\text{terms in Eq. (3.20)}]\right\}. \tag{3.21}$$

Notice that the terms of zero order in k cancel, as they should, making the amplitude proportional to k and the conductivity proportional to k^2. This cancellation depends critically on the closed loop terms in Eq. (3.2). A consistent theory with dynamic screening cannot be made without these terms, except if $\omega=0$, in which case $u_1=u_2$, so that the term of order 1 vanishes identically without cancellation. In a two-component plasma, the terms corresponding to the k-independent terms in Eq. (3.21) and (3.20) are responsible for the leading contribution to the conductivity.

Using Eqs. (3.20) and (3.21), the term proportional to k in the current amplitude can be written

$$-i\hat{e}\cdot\mathbf{J}^c=(\lambda^{3/2}/\omega)V_s^+(q,u_1)V_s^+(q,u_2)\{\hat{e}\cdot\mathbf{k}q^2-4\hat{e}\cdot\mathbf{q}\mathbf{q}\cdot\mathbf{k}+(1/\omega)(\hat{e}\cdot\mathbf{q}\mathbf{k}\cdot\mathbf{p}_1+\mathbf{k}\cdot\mathbf{q}\hat{e}\cdot\mathbf{p}_1)[V_s^+(q,u_1)]^{-1}$$
$$-(1/\omega)(\hat{e}\cdot\mathbf{q}\mathbf{k}\cdot\mathbf{p}_2+\mathbf{k}\cdot\mathbf{q}\hat{e}\cdot\mathbf{p}_2)[V_s^+(q,u_2)]^{-1}+2(\hat{e}\cdot\mathbf{q}\mathbf{q}\cdot\mathbf{k}/q^2)[(u_1/\omega)[V_s^+(q,u_1)]^{-1}$$
$$+(u_2/\omega)[V_s^+(q,u_2)]^{-1}]\}\equiv(\lambda^{3/2}/\omega)V_s^+(q,u_1)V_s^+(q,u_2)\hat{e}\cdot\mathbf{M}. \quad (3.22)$$

$$4\pi e_ie_j\,\mathrm{Im}\sigma_{ij}(k,\omega)=\frac{\lambda}{4\omega^2}(2\pi)^4\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3p_1}{(2\pi)^3}\int\frac{d^3p_2}{(2\pi)^3}\int_{-\infty}^{\infty}du_1\int_{-\infty}^{\infty}du_2\delta(\omega-u_1-u_2)\delta(u_1-\mathbf{q}\cdot\mathbf{p}_1-\tfrac{1}{2}\hbar q^2)$$
$$\times\delta(u_2+\mathbf{q}\cdot\mathbf{p}_2-\tfrac{1}{2}\hbar q^2)e^{-\frac{1}{2}(p_1^2+p_2^2)}|V_s(q,u_1)|^2|V_s(q,u_2)|^2|\hat{e}\cdot\mathbf{M}|^2. \quad (3.23)$$

Longitudinal Conductivity

To obtain the longitudinal conductivity from Eqs. (3.22) and (3.23), let $\hat{e}=\hat{k}$. In the following, let

$$[V_s^+(q,u_1)]^{-1}=[q^2+Q_0^+(u_0/q)]=D(u_1),\quad [V_s^+(q,u_2)]^{-1}=[q^2+Q_0^+(u_2/q)]=D(u_2). \quad (3.24)$$

Then

$$\hat{k}\cdot\mathbf{M}=k\{q^2+(\hat{k}\cdot\hat{q})^2[-4q^2+(2u_1/\omega)D(u_1)+(2u_2/\omega)D(u_2)]$$
$$+(2/\omega)(\hat{k}\cdot\mathbf{q})(\hat{k}\cdot\mathbf{p}_1)D(u_1)-(2/\omega)(\hat{k}\cdot\mathbf{q})(\hat{k}\cdot\mathbf{p}_2)D(u_2)\}. \quad (3.25)$$

Since the final result cannot depend on the direction of k, the average over the directions of $\hat{k}$ can be taken after squaring $(\hat{k}\cdot\mathbf{M})$. The following identities are useful for this:

$$\frac{1}{4\pi}\int d\Omega_k(\hat{k}\cdot\mathbf{a})(\hat{k}\cdot\mathbf{b})=\tfrac{1}{3}\mathbf{a}\cdot\mathbf{b}, \quad (3.26a)$$

$$\frac{1}{4\pi}\int d\Omega_k(\hat{k}\cdot\mathbf{a})^2(\hat{k}\cdot\mathbf{b})(\hat{k}\cdot\mathbf{c})=\frac{1}{15}[a^2(\mathbf{b}\cdot\mathbf{c})+2(\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{c})]. \quad (3.26b)$$

On squaring and averaging over $\hat{k}$ an expression of the following form arises

$$\frac{1}{4\pi}\int d\Omega_k|\hat{k}\cdot\mathbf{M}|^2=A+B(\mathbf{p}_1\cdot\mathbf{p}_2)+Cp_1^2+Dp_2^2, \quad (3.27)$$

where the coefficients A, B, C, and D are functions of q^2, u_1, u_2, $D(u_1)$ and $D(u_2)$. At this point the $\mathbf{p}_1$ and $\mathbf{p}_2$ integrals can be done using the following formulas:

$$\int d^3p_1\int d^3p_2\delta\left(u_1-\mathbf{p}_1\cdot\mathbf{q}-\frac{\hbar q^2}{2}\right)\delta\left(u_2+\mathbf{p}_2\cdot\mathbf{q}-\frac{\hbar q^2}{2}\right)e^{-(p_1^2+p_2^2)}[1;\mathbf{p}_1\cdot\mathbf{p}_2;p_1^2]$$
$$=\frac{(2\pi)^2}{q^2}e^{\hbar\omega}e^{-\hbar^2q^2/4}e^{-(u_1^2+u_2^2)/2q^2}\left[1;-\frac{u_1u_2}{q^2};\frac{u_1^2}{q^2}\right]. \quad (3.28)$$

It is consistent with the assumptions discussed in Sec. II to set $e^{\hbar\omega}=1$ (i.e., if in ordinary units $\beta\hbar\omega\ll1$). However, since the integration is over arbitrarily large values of q, we *cannot*, in general, consider $\hbar^2q^2\ll1$.

Collecting results, Eq. (3.23) becomes

$$4\pi\,\mathrm{Im}\sigma_L(k,\omega)=\frac{1}{120\pi^2}\lambda\frac{k^2}{\omega^4}\int dqq^2e^{-\frac{1}{4}\hbar^2q^2}\int du_1\int du_2\delta(\omega-u_1-u_2)e^{-(u_1^2+u_2^2)/2q^2}|V_s(q,u_1)|^2|V_s(q,u_2)|^2$$
$$\times[23\omega^2q^2+8|D(u_1)|^2+8|D(u_2)|^2]. \quad (3.29)$$

Changing integration variables to

$$u_1=u+\tfrac{1}{2}\omega\,,\quad u_2=-u+\tfrac{1}{2}\omega\,,$$

the delta function is trivially removed and using the definitions of Eq. (3.24) the result is obtained:

$$4\pi\,\mathrm{Im}\sigma_L(k,\omega)=\frac{1}{15\pi^2}\lambda\frac{k^2}{\omega^4}\int dqq^2e^{-\frac{1}{4}\hbar^2q^2}\int due^{-\omega^2/4q^2}e^{-u^2/q^2}$$
$$\times\left[\frac{23}{8}\omega^2q^2|V_s^+(q,\tfrac{1}{2}\omega+u)|^2|V_s^+(q,\tfrac{1}{2}\omega-u)|^2+|V_s^+(q,\tfrac{1}{2}\omega+u)|^2+|V_s^+(q,\tfrac{1}{2}\omega-u)|^2\right].\quad(3.30)$$

It is obvious that the last two terms in brackets make equal contributions. Letting $z=u/q$ the result can be written in terms of two integrals $J(\omega)$ and $I(\omega)$:

$$4\pi\,\mathrm{Im}\sigma_L(k,\omega)=\frac{2}{15\pi^2}\lambda\frac{k^2}{\omega^4}\left(\frac{23}{16}J(\omega)+I(\omega)\right),\quad(3.31)$$

where

$$J(\omega)=\omega^2\int_0^\infty dqq^5e^{-\omega^2/4q^2}e^{-\hbar^2q^2/4}\int_{-\infty}^{\infty}dz\frac{e^{-z^2}}{|q^2+Q_0^+[z+(\omega/2q)]|^2|q^2+Q_0^+[z-(\omega/2q)]|^2},\quad(3.32)$$

$$I(\omega)=\int_0^\infty dqq^3e^{-\omega^2/4q^2}e^{-\hbar^2q^2/4}\int_{-\infty}^{\infty}dz\frac{e^{-z^2}}{|q^2+Q_0^+[z+(\omega/2q)]|^2}.\quad(3.33)$$

Note that in the expression for $J(\omega)$ we can set $\hbar=0$ but that $I(\omega)$ diverges logarithmically in this case. This behavior at large q will be discussed more fully below.

Transverse Conductivity

To obtain the transverse conductivity take $\hat{e}=e^{(1,2)}$ and average over polarizations in Eqs. (3.25) and (3.22) using

$$\tfrac{1}{2}\sum_{\alpha=1,2}(\hat{e}^\alpha\cdot\mathbf{a})(\hat{e}^\alpha\cdot\mathbf{b})=\tfrac{1}{2}[\mathbf{a}\cdot\mathbf{b}-(\mathbf{a}\cdot\mathbf{kk}\cdot\mathbf{b}/k^2)],\quad(3.34)$$

and $\mathbf{k}\cdot\hat{e}^{(1,2)}=0$. Omitting the superscript on $\hat{e}^{(1,2)}$ Eq. (3.22) gives

$$\hat{e}\cdot\mathbf{M}=k\{-4(\hat{e}\cdot\mathbf{q})(\mathbf{q}\cdot\hat{k})-(1/\omega)(\hat{e}\cdot\mathbf{q}\hat{k}\cdot\mathbf{p}_2+\hat{k}\cdot\mathbf{q}\hat{e}\cdot\mathbf{p}_2)D(u_1)$$
$$+(1/\omega)(\hat{e}\cdot\mathbf{q}\hat{k}\cdot\mathbf{p}_1+\hat{k}\cdot\mathbf{q}\hat{e}\cdot\mathbf{p}_1)D(u_2)+(2\hat{e}\cdot\mathbf{qq}\cdot\hat{k}/q^2)[(u_1/\omega)D(u_1)+(u_2/\omega)D(u_2)]\}.\quad(3.35)$$

On squaring this and summing over polarizations according to Eq. (3.34) a sum of terms of the form $(\hat{k}\cdot\mathbf{q})^a(\hat{k}\cdot\mathbf{p}_1)^b\times(\hat{k}\cdot\mathbf{p}_2)^c$ where $a+b+c=2$ or 4 is obtained. The average over the directions of $\hat{k}$ is then performed using Eqs. (3.26), resulting after some algebra in the expression

$$\frac{1}{4\pi}\int d\Omega_k\sum_{\alpha=1,2}|\hat{e}^\alpha\cdot\mathbf{M}|^2=\bar{A}+\bar{B}(\mathbf{p}_1\cdot\mathbf{p}_2)+\bar{C}p_1^2+\bar{D}p_2^2,\quad(3.36)$$

which is the same form as Eq. (3.27). Again using Eq. (3.28) to perform the p_1 and p_2 integrations the result emerges:

$$4\pi\,\mathrm{Im}\sigma_T(k,\omega)=\frac{1}{120\pi^2}\lambda\frac{k^2}{\omega^4}\int dqq^2e^{-\hbar^2q^2/4}\int du_1\int du_2\delta(\omega-u_1-u_2)e^{-(u_1^2+u_2^2)/q^2}|V_s^+(q,u_1)|^2|V_s^+(q,u_2)|^2$$
$$\times[16\omega^2q^2+6|D(u_1)|^2+6|D(u_2)|^2],\quad(3.37)$$

which can again be expressed in terms of the integrals $I(\omega)$ and $J(\omega)$:

$$4\pi\,\mathrm{Im}\sigma_T(k,\omega)=\frac{1}{10\pi^2}\lambda\frac{k^2}{\omega^4}(\tfrac{4}{3}J(\omega)+I(\omega)).\quad(3.38)$$

IV. DISCUSSION OF RESULTS AND APPLICATION TO INCOHERENT SCATTERING

The frequency dependence of our results, Eqs. (3.31) and (3.35), is considerably different than that of previous workers.[1–5] To make contact with the results of classical calculations, we must discuss the treatment of the large-q (or short-range) cutoffs in our results. Strictly speaking, our results are valid only if $kT \gg$rydberg which is the condition for validity of the Born collision approximation in which case the thermal de Broglie wavelength $\hbar(2m/\beta)^{1/2}$ is greater than the classical distance of closest approach $e^2\beta$. Thus the exponential, $\exp[-(\hbar^2/4)q^2]$, in the integrand of Eq. (3.33) enforces a cutoff at approximately the thermal deBroglie wave number. For lower temperatures, $kT \ll$rydberg, one expects the cutoff at the distance of closest approach and this is usually accomplished by dropping the exponential and cutting the q integral off at $q_{\max}=3/e^2\beta=(12\pi/\lambda)k_D$.

Again we note that $J(\omega)$ is sufficiently convergent at $q\rightarrow\infty$ that the cutoff introduces higher order terms in $\hbar$ or λ which must be dropped for consistency in our calculation. However, the cutoff must be kept in $I(\omega)$ since it introduces a logarithmic dependence on $\hbar$ or λ in the two cases. It is probably possible in the electron gas model to treat the large-q cutoff exactly at lower temperatures by summing ladder diagrams of electron-electron collisions, but we shall not attempt this here.

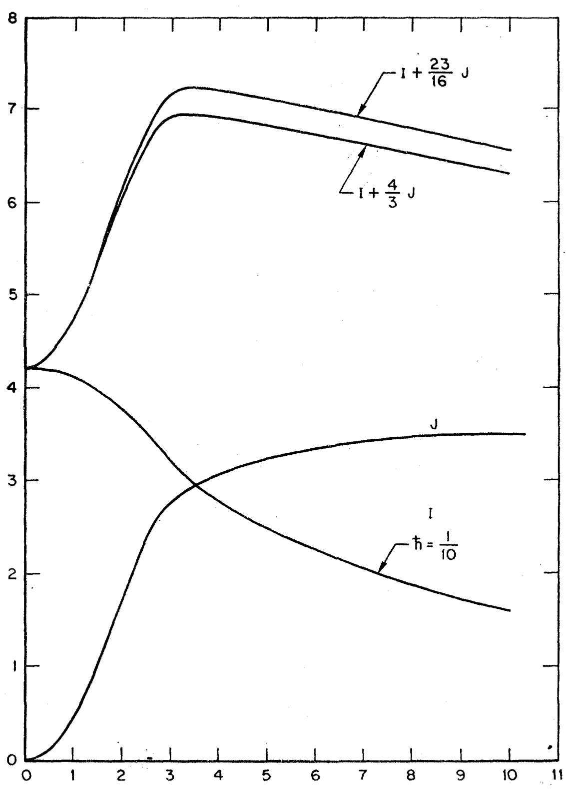

FIG. 3. Numerical evaluation of the integrals I and J for the value $\hbar=0.1$.

The values of $I(\omega)$ and $J(\omega)$ for $\omega\gg 1$ (i.e., $\omega\gg\omega_p$) are easy to obtain for, in this case, the screening becomes ineffective.

$$I(\omega)=\int \frac{dq}{q} e^{-\hbar^2 q^2/4} e^{-\omega^2/4q^2}\int dz\, e^{-z^2} \tag{4.1}$$

and in the limit of small $\hbar$ this becomes

$$I(\omega)\approx\pi^{1/2}\ln(4e^{-C}/\hbar\omega)\,, \tag{4.1a}$$

where $C\approx 0.58$.

To evaluate $J(\omega)$ we change variables and rewrite the integral in the form

$$J(\omega)=\int_0^\infty du\,u e^{-u^2/4}\int_{-\infty}^{\infty} dz \times\frac{e^{-z^2}}{|1+(u^2/\omega^2)Q(z-\frac{1}{2}u)|^2|1+(u^2/\omega^2)Q(z+\frac{1}{2}u)|^2} \tag{4.2}$$

and in the limit $\omega\rightarrow\infty$, we have

$$J(\omega)=2(\pi)^{1/2}.$$

In the low-frequency region ($\omega\ll 1$) the screening complicates the integrals. If we replace the dynamic screening function $Q(y)$ by a static screening constant K^2 (where $K^2\approx 1$), then near $\omega=0$ we have

$$I(\omega)\approx\int dq\,q^3 e^{-\hbar^2q^2/4}\int dy\frac{e^{-y^2}}{(q^2+K^2)^2}, \tag{4.3}$$

$$J(\omega)\approx\omega^2\int dq\,q^5\int dy\frac{e^{-y^2}}{(q^2+K^2)^4}. \tag{4.4}$$

We now obtain

$$I(\omega)\approx\frac{(\pi)^{1/2}}{2}\left(\int_b^\infty dx\frac{e^{-x}}{x}-1\right), \tag{4.4a}$$

where $b=\frac{1}{4}\hbar^2K^2$, and in the limit of $\hbar\rightarrow 0$ we get

$$I(\omega)\approx\tfrac{1}{2}(\pi)^{1/2}(-\ln(e^C\tfrac{1}{4}\hbar^2K^2)-1) \tag{4.4b}$$

$$\approx(\pi)^{1/2}(\ln(0.67\hbar)-\tfrac{1}{2}) \tag{4.4c}$$

since $K\approx 1$. The other integral yields, near $\omega=0$,

$$J(\omega)\approx(\pi^{1/2}/6)(\omega^2/K^2). \tag{4.4d}$$

Note that $J(\omega)$ vanishes like ω^2 in this limit.

In Fig. 3 the results of numerical integration for $J(\omega)$ and $I(\omega)$ are plotted for the values $\hbar=10^{-1}$. The results obtained using the classical high wave number cutoff do not differ significantly from these for typical values of λ because of the weak logarithmic dependence of $I(\omega)$ on the cutoff. For $\omega\ll 1$ we notice that the contribution of $J(\omega)$ is negligible compared to $I(\omega)$ but at $\omega\sim 2$ the two terms are comparable and for $\omega\gg 1$ $J(\omega)$ dominates. It is not difficult to show that the Balescu type[7,8] of kinetic equation yields a result which is equivalent to replacing the combinations $I(\omega)+(23/16)\times J(\omega)$ and $I(\omega)+\frac{3}{2}J(\omega)$ in Eqs. (3.31) and (3.38) by simply $I(0)$. For $\omega>1$ we see this is a poor approximation. The integral $J(\omega)$ which contains the primary effect of the perturbation of the screening electrons by the incident field is the *dominant* contribution for $\omega\gtrsim 2$.

It is probably meaningful to associate the enhancement of the curve in Fig. 3 due to $J(\omega)$ with the production of two plasmons in the final state as suggested by Figs. 2(c) and 2(d). It is clear from Eq. (3.32) that only $J(\omega)$ contains a contribution from the collective resonance associated with the dynamically screened interaction in *both* of the outgoing lines. This enhancement, however, cannot be described as a resonance. From Fig. 3 we see that it is very broad. This is due to the dispersion and Landau damping of the plasmons which contribute for larger values of q in the integrand of (3.32). Notice that small values of q, for which the plasmon resonance is sharp, are suppressed by the factor of q^5 in the integrand. Because of the $1/\omega^4$ factor of Eqs. (3.31) and (3.38) the complete conductivity be a smoothly decreasing curve even for $\omega\gtrsim 2\omega_p$ and the enhancement discussed above will be hardly discernable.

We can use our result to compare the damping of plasma oscillations due to electron-electron collisions with the work of other authors. We will make an explicit comparison to the recent work of Wu and Klevans[3] who make comparisons with Refs. 1 and 2. The damping rate defined by Wu and Klevans is

$$\gamma/\omega_p=2\pi\,\mathrm{Im}\sigma_L(k,\omega_p)$$

in our notation (and therefore differs by a factor of 2 from the conventions in Ref. 9). Using Eq. (3.31) we have the result (in plasma units)

$$\frac{\gamma}{\omega_p}=\frac{1}{5\pi^2}\lambda k^2\left[I(1)+\frac{23}{16}J(1)\right]$$
$$=\frac{1}{15\pi^{3/2}}\left(\frac{k}{k_D}\right)^2\frac{k_D{}^3}{n}[2.95]. \qquad (4.6)$$

The Wu and Klevans result including only electron-electron correlations [the last terms on the right in their Eqs. (50b) and (50c)] is

$$\gamma/\omega_p=\frac{1}{15\pi^{3/2}}\left(\frac{k}{k_D}\right)^2\frac{k_D{}^3}{n}\left[\ln\left(0.707\frac{k_T}{k_D}\right)+\ln(1870)\right]. \qquad (4.7)$$

In their expression we have used the quantum cutoff $k_T=k_D/\beta\hbar\omega_p$ (instead of their k_L) to make contact with our results. For the value $k_T/k_D=1/(\beta\hbar\omega_p)=10$ for which our result was obtained the factor in square brackets in Eq. (4.7) is 9.49. Our result which should be exact in the limit as $k_D{}^3/n\rightarrow 0$ for $k\ll k_D$ and for $kT\gg$rydberg for which the Born approximation is valid is more than a factor of three smaller than this. This may be accounted for by their approximate treatment of the short-range cutoff. In the case $kT\ll$rydberg for which the classical cutoff applies, neither method handles the cutoff exactly and we can only say that the results consistent to within the uncertainties in the cutoff procedure. It is interesting to note that a very simple calculation of this conductivity neglecting collective screening effects entirely yields the factors [3.13] for $k_T/k_D=10$ which is surprisingly close to the exact result in Eq. (4.6). [This approximation corresponds to using the high ω approximation Eq. (4.1a) for $I(\omega)$ and neglecting $J(\omega)$ entirely!]

The electron gas result which we are considering is actually applicable near $2\omega_p$ to a two-component plasma in the limit of infinite ion mass. The contributions to the conductivity due to electron-ion and ion-ion collisions are smooth near $\omega=2\omega_p$. This follows since the primary change in the formulas in these cases is to replace the exponential $\exp[-(u_1{}^2+u_2{}^2)/2q^2]$ in Eq. (3.29) to $\exp[-(M_1u_1{}^2+M_2u_2{}^2)/2q^2]$, where M_1 and M_2 are the masses in units of m.[14] Thus, near the resonance where u_1 and u_2 are near unity (i.e., ω_p), these contributions are $\exp[-(M_1+M_2-2)/q^2]$ smaller than the contribution from electron-electron collisions. However, $e-i$, $e-e$, and $i-i$ collisions contribute to the smooth background of thermal noise at $\omega\approx 2\omega_p$ and of these the $e-i$ contribution, which is finite at $k\rightarrow 0$, is the major contribution. It is also straightforward to show that the effect of ion screening in electron-electron collisions is negligible in the limit $m/M\rightarrow 0$.

We can apply this result to the observation of the frequency shifts at the second harmonic of ω_p in the incoherent scattering of radiation from plasmas. It follows from the results of Ref. 11 that the spectral distribution of the scattered light is essentially proportional to $\mathrm{Im}\sigma_L(k,\omega)$. Near $2\omega_p$ the only contribution from two plasmon excitation is the weak inflection of the $e-e$ contribution which is superimposed on the smooth background of noise contributed by the $e-i$ contribution which is larger by a factor of order k^{-2}. The inflection would be at most a one-percent effect (say for $k\geq 10^{-1}$) and is most likely unobservable.

For a plasma in which appreciable density gradients exist over distances short compared to a Debye length the production of the second harmonic will be much stronger as can be seen from the work of Boyd,[15] for example. However, for the incoherent scattering of radiation from the ionosphere where such strong density gradients do not exist our calculation shows that a frequency shift at about $\pm 2\omega_p$ is probably unobservable.

[14] The calculation in the arbitrary mass case follows essentially exactly that in Sec. III. The k^2 correction due to electron-ion collisions has recently been computed by H. L. Berk, Phys. Fluids **7**, 257 (1964). This calculation coupled with our result gives the complete collisional conductivity to order k^2.

[15] T. J. M. Boyd, Phys. Fluids **7**, 59 (1964).

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APPENDIX: RULES FOR EVALUATING DIAGRAMS

(0) Draw all topologically distinct open diagrams for the current amplitudes $J_i(\alpha,\beta;k,\omega)$ leading from the initial state of one quantum $(\mathbf{k},\omega)$ to a given excited final state of the system (examples are shown in Figs. 1 and 2). The current amplitude has a contribution from each diagram obtained by multiplying together the following factors:

(1) To each internal particle line carrying momentum p and energy p_0 there corresponds a factor of

$$i\tilde{G}(p,p_0)=i/(p_0-\xi_p), \tag{A1}$$

where $\xi_p=(p^2/2m)-\mu$, where m is the particle's mass in units of the electron mass and μ is the chemical potential for this type of particle. In natural units μ is defined by

$$e^\mu=(\hbar^3/\lambda)(2\pi/m)^{3/2}. \tag{A2}$$

(2) To each Coulomb interaction line carrying momentum $\hbar\mathbf{q}$ and energy $\hbar q_0$ there corresponds a factor of the dynamically screened Coulomb propagator

$$-i\tilde{V}_s(\hbar q,\hbar q_0)=\frac{-i}{q^2+\tilde{Q}(\hbar q,\hbar q_0)}, \tag{A3}$$

where $\tilde{Q}(\hbar q,\hbar q_0)$ is the screening function (i.e., the proper polarization part). To lowest order in λ (i.e., the RPA) $\tilde{Q}$ is given by

$$\tilde{Q}(\hbar q,\hbar q_0)=\lambda\int\frac{d^3p}{(2\pi\hbar)^3}\frac{f(p-(\hbar/2)\mathbf{q})-f(p+(\hbar/2)\mathbf{q})}{\hbar\omega-\hbar(\mathbf{p}\cdot\mathbf{q}/m)}, \tag{A4}$$

where m is the mass of the screening particles in electron units. In the classical $(\hbar\to 0)$ limit this can be written

$$\lim_{\hbar\to 0}\tilde{Q}_0(q,q_0)=\tilde{Q}_0(mq_0/q)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}dy\frac{y}{y-m(q_0/q)}e^{-y^2/2} \tag{A5}$$

so in this limit

$$\lim_{\hbar\to 0}\tilde{V}_s(\hbar q,\hbar q_0)=\tilde{V}_s(q,q_0)=\frac{1}{q^2+\tilde{Q}_0(mq_0/q)}.$$

(3) At each Coulomb vertex there is a factor

$$iz(\lambda)^{1/2} \tag{A6}$$

where z is the charge (in electron units) of the particle line.

(4) To each single photon-particle vertex there corresponds a factor

$$-i(\lambda)^{1/2}z\hat{e}\cdot(1/2m)(\mathbf{p}+\mathbf{p}'), \tag{A7}$$

where $\hat{e}$ is the (transverse) photon polarization and $\mathbf{p}$ and $\mathbf{p}'$ are the incoming and outgoing particle momenta.

(5) There is a factor of (-1) for each closed loop.

(6) Energy and momentum are conserved at each vertex and internal momentum and energy variables are summed over according to

$$i\sum_{p_0}\int\frac{d^3p}{(2\pi\hbar)^3}\quad\text{or}\quad i\sum_{q_0}\frac{d^3q}{(2\pi)^3}, \tag{A8}$$

where the energy variables take on the discrete values p_0, $q_0=i\pi\nu$, where ν runs over odd integers for fermions and even integers for bosons (including the screened interaction).

(7) Add all current amplitudes with the same initial and final states to insert into Eq. (2.3). Diagrams which differ only the exchange of identical fermions in the final state differ by a factor of (-1). The sum and average over initial and final states is accomplished by inserting weight factors $[1-f(p)]$ for particles and $f(p)$ for holes where

$$f(p)=[e^{\beta\xi_p}\pm 1]^{-1}. \tag{A9}$$

In the limit of classical statistics

$$f(p)=e^{-\xi_p}=e^{\mu}e^{-(p^2/2m)}=(\hbar^3/\lambda)(2\pi/m)^{3/2}e^{-(p^2/2m)}\ll 1 \tag{A10}$$

all final particle and hole states are summed using the familiar $(2\pi\hbar)^{-3}\int d^3p$. Since this counts the same final state more than once when identical particles are involved the result must be divided by $(n!)$, where n is the number of particles of a given type in the final state. [Thus, in Eq. (3.1) below there occurs a factor of $\frac{1}{2}$.]

(8) The amplitudes must be analytically continued to continuous values of the external energy variables by setting $q_0=u+i\epsilon$ and $k_0=\omega+i\epsilon$, where ϵ is a positive infinitesimal.